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Logarithmic forms and anti-invariant forms of reflection groups

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Dedicated to Peter Orlik on his sixtieth birthday

Abstract.

Let W be a finite group generated by unitary reflections and $\mathcal{A}$ be the set of reflecting hyperplanes. We will give a characterization of the logarithmic differential forms with poles along $\mathcal{A}$ in terms of anti-invariant differential forms. If W is a Coxeter group defined over $\mathbf{R}$, then the characterization provides a new method to find a basis for the module of logarithmic differential forms out of basic invariants.

Basic definitions. Let V be an ℓ-dimensional unitary space. Let $W \subset \mathbf{GL}(V)$ be a finite group generated by unitary reflections and $\mathcal{A}$ be the set of reflecting hyperplanes. We say that W is a *unitary reflection group* and $\mathcal{A}$ is the corresponding *unitary reflection arrangement*. Let S be the algebra of polynomial functions on V. The algebra S is naturally graded by $S = \bigoplus_{q \geq 0} S_q$ where S_q is the space of homogeneous polynomials of degree q. Thus $S_1 = V^*$ is the dual space of V. Let Der_S be the S-module of $\mathbf{C}$-derivations of S. We say that $\theta \in \mathrm{Der}_S$ is homogeneous of degree q if $\theta(S_1) \subseteq S_q$. Choose for each hyperplane $H \in \mathcal{A}$ a linear form $\alpha_H \in V^*$ such that $H = \ker(\alpha_H)$. Define $Q \in S$ by

$$Q = \prod_{H \in \mathcal{A}} \alpha_H \, .$$

The polynomial Q is uniquely determined, up to a constant multiple, by the group W. When convenient we choose a basis $e_1, \ldots, e_l$ for V and let $x_1, \ldots, x_l$ denote the dual basis for V^*. Let $\langle \ , \ \rangle : V^* \times V \to \mathbf{C}$ denote the natural pairing. Thus $\langle x_i, e_j \rangle = \delta_{ij}$. For each $v \in V$ let $\partial_v \in \mathrm{Der}_S$ be the unique derivation such that $\partial_v x = \langle x, v \rangle$ for $x \in V^*$. Define $\partial_i \in \mathrm{Der}_S$ by $\partial_i = \partial_{e_i}$. Then $\partial_i x_j = \delta_{ij}$ and Der_S is a free S-module with basis $\partial_1, \ldots, \partial_l$. There is a natural isomorphism $S \otimes V \to \mathrm{Der}_S$ of S-modules given by

$$f \otimes v \mapsto f \partial_v$$

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for $f \in S$ and $v \in V$. Let $\Omega^1 = \mathrm{Hom}_S(\mathrm{Der}_S, S)$ be the S-module dual to Der_S. Define $d : S \to \Omega^1$ by $df(\theta) = \theta(f)$ for $f \in S$ and $\theta \in \mathrm{Der}_S$. Then $d(ff') = (df)f' + f(df')$ for $f, f' \in S$. Furthermore, Ω^1 is a free S-module with basis $dx_1, \ldots, dx_l$ and $df = \sum_{i=1}^{l} (\partial_i f) dx_i$. There is a natural isomorphism $S \otimes V^* \to \Omega^1$ of S-modules given by

$$f \otimes x \mapsto f dx$$

for $f \in S$ and $x \in V^*$. The modules Der_S and Ω^1 inherit gradings from S which are defined by $\deg(f\partial_v) = \deg(f)$ and $\deg(f dx) = \deg(f)$ if $f \in S$ is homogeneous. Define $\Omega^p = \bigwedge_S^p \Omega^1$ $(p = 1, \ldots, \ell)$. Let $\Omega^0 = S$. The S-module Ω^p is free with a basis $\{dx_{i_1} \wedge \cdots \wedge dx_{i_p} \mid 1 \le i_1 < \cdots < i_p \le \ell\}$. It is naturally isomorphic to $S \otimes_{\mathbf{C}} \bigwedge^p V^*$. Let $\Omega^p(\mathcal{A})$ be the S-module of *logarithmic p-forms* with poles along $\mathcal{A}$ [Sai3][OrT]:

$$\Omega^p(\mathcal{A}) = \{\frac{\eta}{Q} \mid \eta \in \Omega^p, d(\frac{\eta}{Q}) \in \frac{1}{Q}\Omega^{p+1}\}$$

where d is the exterior differentiation.

The unitary reflection group W acts contragradiently on V^* and thus on S. The modules Der_S and Ω^p $(p = 0, \ldots, \ell)$ also have W-module structures so that the above isomorphisms are W-module isomorphisms. If M is an $\mathbf{C}[W]$-module let $M^W = \{x \in M \mid wx = x \text{ for all } w \in W\}$ denote the space of invariant elements in M. Let $M^{\det^{-1}} = \{x \in M \mid wx = \det(w)^{-1}x \text{ for all } w \in W\}$ denote the space of anti-invariant elements in M. Let $R = S^W$ be the invariant subring of S under W. By a theorem of Shephard, Todd, and Chevalley [Bou, V.5.3, Theorem 3] there exist algebraically independent homogeneous polynomials $f_1, \ldots, f_l \in R$ such that $R = \mathbf{C}[f_1, \ldots, f_l]$. They are called *basic invariants*. Elements of $S^{\det^{-1}}$ and $(\Omega^p)^{\det^{-1}}$ are called *anti-invariants* and *anti-invariant p-forms* respectively. It is well-known that $S^{\det^{-1}} = RQ$.

The main theorem. The following theorem gives the relationship between logarithmic forms and anti-invariant forms.

Theorem 1. *For* $0 \le p \le \ell$,

$$\Omega^p(\mathcal{A}) = \frac{1}{Q}(\Omega^p)^{\det^{-1}} \otimes_R S.$$

Proof. When $p = 0$, the result follows from the formula $S^{\det^{-1}} = RQ$. Let $p > 0$. Let $x_1, \ldots, x_\ell$ be an orthonormal basis for V^*. Let $\theta_1, \ldots, \theta_\ell$ be an R-basis for Der_S^W. Then, by [OrT, Theorem 6.59],

$\theta_1, \ldots, \theta_\ell$ is known to be an S-basis for the module $D(\mathcal{A})$ of $\mathcal{A}$-derivations, where

$$D(\mathcal{A}) = \{\theta \in \mathrm{Der_S} \mid \theta(Q) \in QS\}.$$

By the contraction $\langle\ ,\ \rangle$ of a 1-form and a derivation, the S-modules $D(\mathcal{A})$ and $\Omega^1(\mathcal{A})$ are S-dual each other [Sai3, p.268] [OrT, Theorem 4.75] . Let $\{\omega_1, \ldots, \omega_\ell\} \subset \Omega^1(\mathcal{A})$ be dual to $\{\theta_1, \ldots, \theta_\ell\}$. In other words, $\langle\theta_i, \omega_j\rangle = \delta_{ij}$ (Kronecker's delta). Then $\{\omega_1, \ldots, \omega_\ell\}$ is an S-basis for $\Omega^1(\mathcal{A})$. Then each ω_i is obviously W-invariant and

$$\omega_i \in (\frac{1}{Q}\Omega^1)^W = \frac{1}{Q}(\Omega^1)^{\det^{-1}}.$$

Therefore we have

$$\Omega^1(\mathcal{A}) \subseteq \frac{1}{Q}(\Omega^1)^{\det^{-1}} \otimes_R S.$$

By [OrT, Proposition 4.81], the set $\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \mid 1 \le i_1 < \cdots < i_p \le \ell\}$ is a basis for $\Omega^p(\mathcal{A})$. In particular, $\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \in \frac{1}{Q}\Omega^p$. Since $\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}$ is W-invariant, $Q(\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}) \in (\Omega^p)^{\det^{-1}}$. This shows that

$$\Omega^p(\mathcal{A}) \subseteq \frac{1}{Q}(\Omega^p)^{\det^{-1}} \otimes_R S.$$

Conversely let $\omega \in \frac{1}{Q}(\Omega^p)^{\det^{-1}}$. Then $Q\omega \in \Omega^p \subseteq \Omega^p(\mathcal{A})$. Thus $Q\omega$ can be uniquely expressed as

$$Q\omega = \sum_{i_1 < \cdots < i_p} f_{i_1 \cdots i_p} \omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \ (f_{i_1 \cdots i_p} \in S).$$

Act $w \in W$ on both sides, and we get

$$\det(w)^{-1} Q\omega = w(Q)\omega = \sum_{i_1 < \cdots < i_p} w(f_{i_1 \cdots i_p}) \omega_{i_1} \wedge \cdots \wedge \omega_{i_p}.$$

Therefore, by the uniqueness of the expression, we have

$$\det(w)^{-1} f_{i_1 \cdots i_p} = w(f_{i_1 \cdots i_p}) \ (w \in W)$$

and $f_{i_1 \cdots i_p} \in S^{\det^{-1}} = RQ$. This implies that each $f_{i_1 \cdots i_p}/Q$ lies in S and that

$$\omega = \sum_{i_1 < \cdots < i_p} \left(\frac{f_{i_1 \cdots i_p}}{Q}\right) \omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \in \Omega^p(\mathcal{A}).$$

Thus we have shown the inclusion

$$\frac{1}{Q}(\Omega^p)^{\det^{-1}} \otimes_R S \subseteq \Omega^p(\mathcal{A}).$$

Q.E.D.

Taking the W-invariant parts of the both sides in Theorem 1, we have

Corollary 2. *For* $0 \le p \le \ell$,

$$(\Omega^p(\mathcal{A}))^W = \frac{1}{Q}(\Omega^p)^{\det^{-1}}.$$

The following theorem is a special case of a theorem obtained by Shepler [She1].

Theorem 3 (Shepler)**.** *For* $0 \le p \le \ell$,

$$(\Omega^p)^{\det^{-1}} = Q^{1-p} \bigwedge_R^p (\Omega^1)^{\det^{-1}}.$$

Proof. Let $p = 0$. We naturally interpret the "empty exterior product" to be equal to the coefficient ring. Thus the result follows from the formula $S^{\det^{-1}} = RQ$. Let $p > 0$. In the proof of Theorem 1, we have already shown that the both sides have the same R-basis

$$\{Q(\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}) \mid 1 \le i_1 < \cdots < i_p \le \ell\}.$$

Q.E.D.

The Coxeter case. From now on we assume that W is a *Coxeter group*. In other words, for an ℓ-dimensional Euclidean space V, $W \subset \mathbf{GL}(V)$ is a finite group generated by orthogonal reflections and W acts irreducibly on V. The objects like S, R, and Ω^p are defined over $\mathbf{R}$. Note that $\det(w)$ is either $+1$ or -1 for any $w \in W$ and thus $\det = \det^{-1}$.

Recall the definition of the $\mathbf{R}$-linear map $\hat{d} : S \longrightarrow \Omega^1$ in [SoT, Proposition 6.1]:

$$\hat{d}f = \sum_{i=1}^{\ell} (\partial_i f) d(Q(Dx_i)).$$

Here D is a Saito derivation introduced in [Sai2][SYS]. The following proposition is Proposition 6.1 in [SoT]:

Proposition 4 (Solomon-Terao). *Let $f_1, \dots, f_\ell$ be basic invariants. Then*

$$(\Omega^1)^{\det} = R\hat{d}f_1 \oplus \cdots \oplus R\hat{d}f_\ell.$$

From Theorem 3 and Proposition 4 we get

Corollary 5. *For $0 \le p \le \ell$,*

$$(\Omega^p)^{\det} = \bigoplus_{1 \le i_1 < \cdots < i_p \le \ell} RQ^{1-p}(\hat{d}f_{i_1} \wedge \cdots \wedge \hat{d}f_{i_p}).$$

Using Theorem 1, we have

Corollary 6. *For $0 \le p \le \ell$,*

$$\Omega^p(\mathcal{A}) = \bigoplus_{1 \le i_1 < \cdots < i_p \le \ell} SQ^{-p}(\hat{d}f_{i_1} \wedge \cdots \wedge \hat{d}f_{i_p}).$$

This corollary gives a new method using the new differential operator $\hat{d}$ to calculate a basis for the module of logarithmic forms.

Taking the W-invariant parts of the both sides in Corollary 6, we also have

Corollary 7. *For $0 \le p \le \ell$,*

$$(\Omega^p(\mathcal{A}))^W = \bigoplus_{1 \le i_1 < \cdots < i_p \le \ell} RQ^{-p}(\hat{d}f_{i_1} \wedge \cdots \wedge \hat{d}f_{i_p}).$$

Example 8 (B_2). When W is the Coxeter group of type B_2, we can choose

$$f_1 = \frac{1}{2}(x_1^2 + x_2^2),\ f_2 = \frac{1}{4}(x_1^4 + x_2^4).$$

Then, as seen in [SoT, §5.2], the operator $\hat{d}$ in Proposition 4 satisfies

$$\hat{d}x_1 = -dx_2,\ \hat{d}x_2 = dx_1.$$

Thus

$$\hat{d}f_1 = -x_1 dx_2 + x_2 dx_1,\ \hat{d}f_2 = -x_1^3 dx_2 + x_2^3 dx_1.$$

Then $\hat{d}f_1$ and $\hat{d}f_2$ form an R-basis for $(\Omega^1)^{\det}$ and $\hat{d}f_1/Q$ and $\hat{d}f_2/Q$ form an S-basis for $\Omega^1(\mathcal{A})$ as Corollaries 5 and 6 assert.

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